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OPTIMUM BRAKING OF ELASTIC OBJECT

Abstract: In [1] the classical design procedure of elastic suspension of solid body is given. It includes static calculation of suspension (definition of static reactions); calculation of own frequencies of vibrating protection system with six degrees of freedom; definition of amplitudes of the object compelled fluctuations at harmonious influence. In [2] results of numerical experiments on research of elastic suspension dynamics, executed on the basis of specially developed program "VIBRO", are given. In [3, 4] laws of optimum motion of elastic systems with final number of degrees of freedom are proved and illustrated on examples.
The research purpose – the analysis of elastic suspension dynamics at optimum braking.

1. Differential equations of dynamics of elastic suspension relative movement

The equations of motion of elastic suspension are made, using the Dalamber principle. For not free system the geometrical sum of the main vectors of set forces, support reactions and inertial forces is equal to zero and the geometrical sum of the main moments of set forces, support reactions and inertial forces concerning to the center is equal to zero:

$$\overline{P}^* + \overline{R}^* + \overline{\Phi}^* = 0, \qquad \overline{M}_o^P + \overline{M}_o^R + \overline{M}_o^{\Phi} = 0, \tag{1}$$

where $\overline{P}^*$ – the main vector of set forces; $\overline{R}^*$ – main vector of support reactions (shock-absorbers); $\overline{\Phi}^*$ – main vector of inertial forces; $\overline{M}_o^P$ – the main moment of set forces concerning to the center O; $\overline{I}_{\hat{\imath}}^{R}$ – main moment of reactions; $\overline{M}_o^{\Phi}$ – main moment of inertial forces.

The object participates in two motions: object motion relatively to mobile system of coordinates is called as relative motion, and object motion together with mobile system of coordinates – transfer movement.

After designing on the axis of mobile system of coordinates (figure 1) from (1) follows the system of the linear differential equations describing relative movement; it is made under condition of small movements: $sin\theta_x \cong \theta_x$, $cos\theta_x \cong 1$, $sin\theta_y \cong \theta_y$, $cos\theta_y \cong 1$, $sin\theta_z \cong \theta_z$, $cos\theta_z \cong 1$.

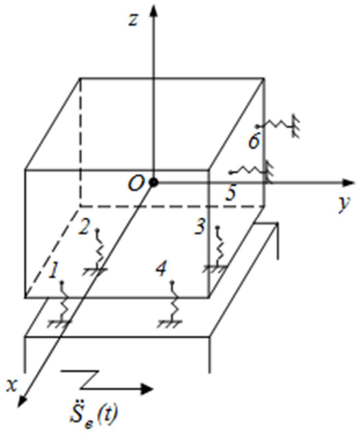

Fig.1. Scheme of the object elastic suspension; 1– 6 –shock-absorbers

For example, movement of a fastening point of the first shock-absorber on the axis y is equal to forward movement of the mass center of a solid body on this axis plus the movements, caused by rotation round axes and so (according to a superposition principle for linear system):

$$v_1 = v + v_1\theta_x + v_1\theta_z\,, \tag{2}$$

where v – movement of the mass center; $v_1^{\theta_x} = l_1 \sin\theta_x \cos\beta_1$; $\cos\beta_1 = \dfrac{Z_1}{l_1}$; $v_1^{\theta_z} = l_1^* \sin\theta_z \cos\beta_1$; $\cos\beta_1^* = \dfrac{Y_1}{l_1^*}$; $\sin\theta_x \approx \theta_x$; $\sin\theta_z \approx \theta_z$. After transformations from (2) dependence equation: $v_1 = v + Z_1\theta_x + Y_1\theta_z$. In the same way movements of fastening points of other shock-absorbers are calculated.

After multiplication of the corresponding rigidity factors of support on movements and summation of the received increments of restoring forces, the expressions for restoring forces, which enter into the right parts (considering signs) of the motion differential equations, are received. Equations of linear system motion:

$$\begin{cases} m\dfrac{d^2u}{dt^2} = -C_u u - C_{uy}\theta_y - C_{uz}\theta_z - m\ddot{\xi}_x, \\ m\dfrac{d^2v}{dt^2} = -C_v v - C_{vx}\theta_x - C_{vz}\theta_z - m\ddot{\xi}_y, \\ m\dfrac{d^2w}{dt^2} = -C_w w - C_{wx}\theta_x - C_{wy}\theta_y - m\ddot{\xi}_z, \\ I_x\dfrac{d^2\theta_x}{dt^2} = -C_{vx}v - C_{wx}w - D_x\theta_x - D_{xy}\theta_y - D_{xz}\theta_z - B_x, \\ I_y\dfrac{d^2\theta_y}{dt^2} = -C_{uy}u - C_{wy}w - D_{xy}\theta_x - D_y\theta_y - D_{yz}\theta_z - B_y, \\ I_z\dfrac{d^2\theta_z}{dt^2} = -C_{uz}u - C_{vz}v - D_{xz}\theta_x - D_{yz}\theta_y - D_z\theta_z - B_z, \end{cases} \quad , \tag{3}$$

where m – mass of object; ; I_x, I_y, I_z – the inertial moments; C_u, C_{uy}… – rigidity factors.

If, for example, to exclude shock-absorbers №5 and №6 and rigidity in the axis z $C_{wi} = 0$, the expressions for rigidity factors (figure 1) are write down so

$$C_u = \sum_{i=1}^{4} C_{ui}\ ;\ C_v = \sum_{i=1}^{4} C_{vi}\ ;\ C_{uy} = \sum_{i=1}^{4} C_{ui} z_i\ ;\ C_{uz} = -\sum_{i=1}^{4} C_{ui} y_i\ ;\ C_{vx} = -\sum_{i=1}^{4} C_{vi} z_i\ ;\ C_{vz} = \sum_{i=1}^{4} C_{vi} x_i\ ;$$

$$D_z = \sum_{i=1}^{4} (C_{ui} y_i^2 + C_{vi} x_i^2);\ D_y = \sum_{i=1}^{4} C_{ui} z_i^2\ ; D_x = \sum_{i=1}^{4} C_{wi} y_i^2\ ; D_{xz} = -\sum_{i=1}^{4} C_{vi} x_i z_i\ ; D_{yz} = -\sum_{i=1}^{4} C_{ui} y_i z_i$$

,

where x_i, y_i, z_i – coordinates of fastening points of shock-absorbers.

2. Justification of the optimum braking law of elastic object

The elastic system (figure 2) at its braking is necessary to move in time, which to multiply period of own fluctuations. At the end of motion the speed and movements of elastic system in relative motion and the speed in transfer motion should be equal to zero.

For the set acceleration of transfer motion (and its two integrals) with the accounting of regional conditions the system of the linear equations for definition C_1, C_2, a, L is received:

$$N_1 := -\frac{2a}{3p} + C_1 - V_0 = 0; \qquad N_2 := \frac{a\left(-\frac{1}{3}\sin^2(pT)\cos(pT) - \frac{2}{3}\cos(pT) \right)}{p} + C_1 = 0;$$

$$N_3 := \frac{a\left(-\frac{1}{9}\frac{\sin^3(pT)}{p} - \frac{2}{3}\frac{\sin(pT)}{p} \right)}{p} + C_1 T + C_2 - L = 0; \quad N_4 := C_2. \tag{4}$$

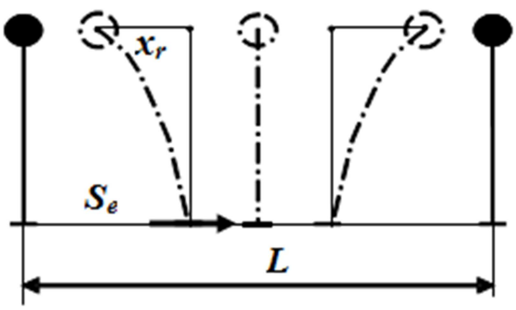

Fig.2. Optimum braking of elastic system with one degree of freedom

The roots of system (4) are found.

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> solve({N1,N2,N3,N4},{C1,a,L,C2});
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$$\{L=\frac{1}{3}\,\frac{V0\,(-\sin(p\,T)^3-6\,\sin(p\,T)+3\,\cos(p\,T)\,T\,p\,\sin(p\,T)^2+6\,\cos(p\,T)\,T\,p)}{p\,(\sin(p\,T)^2\cos(p\,T)+2\,\cos(p\,T)-2)},\ C2=0,$$
$$C1=\frac{\cos(p\,T)\,V0\,(\sin(p\,T)^2+2)}{\sin(p\,T)^2\cos(p\,T)+2\,\cos(p\,T)-2},\ a=\frac{3\,V0\,p}{\sin(p\,T)^2\cos(p\,T)+2\,\cos(p\,T)-2}\} \tag{5}$$

Now, if to accept $p = \pi/T$, then: $L=\frac{V_0T}{2};\quad C_1=\frac{V_0}{2};\quad a=-\frac{3V_0\pi}{4T};\quad C_2=0.$

Considering the found constants, the optimum braking (basis movement, speed and acceleration) assume the following form:

$$S_e=-\frac{3}{4}V_0\left(-\frac{1}{9}\frac{\sin^3(\pi t/T)T}{\pi}-\frac{2}{3}\frac{T\sin(\pi t/T)}{\pi}\right)+\frac{V_0t}{2}+C_2;$$
$$V_e=-\frac{3}{4}V_0\left(-\frac{1}{3}\sin^2(\pi t/T)\cos(\pi t/T)-\frac{2}{3}\cos(\pi t/T)\right)+\frac{V_0}{2}; \tag{6}$$
$$\ddot{S}_e=-\frac{3V_0}{4}\frac{\pi\sin^3(\pi t/T)}{T}.$$

The differential equation describing the optimum relative motion (fluctuations) of object:

$$\frac{d^2x_r}{dt^2}+k^2x_r=\frac{3V_0}{4}\frac{\pi\sin^3(\pi t/T)}{T}. \tag{7}$$

The solution of the equation (7) under initial conditions $x_r(0)=0$, $\dot{x}_r(0)=0$:

$$\mathrm{x}(t)=\frac{9}{2}\,\frac{\sin(k\,t)\,V0\,\pi^4}{k\,(k^4\,T^4-10\,k^2\,T^2\,\pi^2+9\,\pi^4)}$$
$$+\frac{3\,V0\,\pi\,T\left(3\,k^2\,T^2\sin\left(\frac{\pi\,t}{T}\right)-k^2\,T^2\sin\left(\frac{3\,\pi\,t}{T}\right)+\pi^2\sin\left(\frac{3\,\pi\,t}{T}\right)-27\,\pi^2\sin\left(\frac{\pi\,t}{T}\right)\right)}{16\,k^4\,T^4-160\,k^2\,T^2\,\pi^2+144\,\pi^4} \tag{8}$$

Schedules of relative (figure 3) and transfer (figure 4) motions are constructed at basic data: k = 8π; T1 = 2π/k; T = 10T1; V0 = 5.

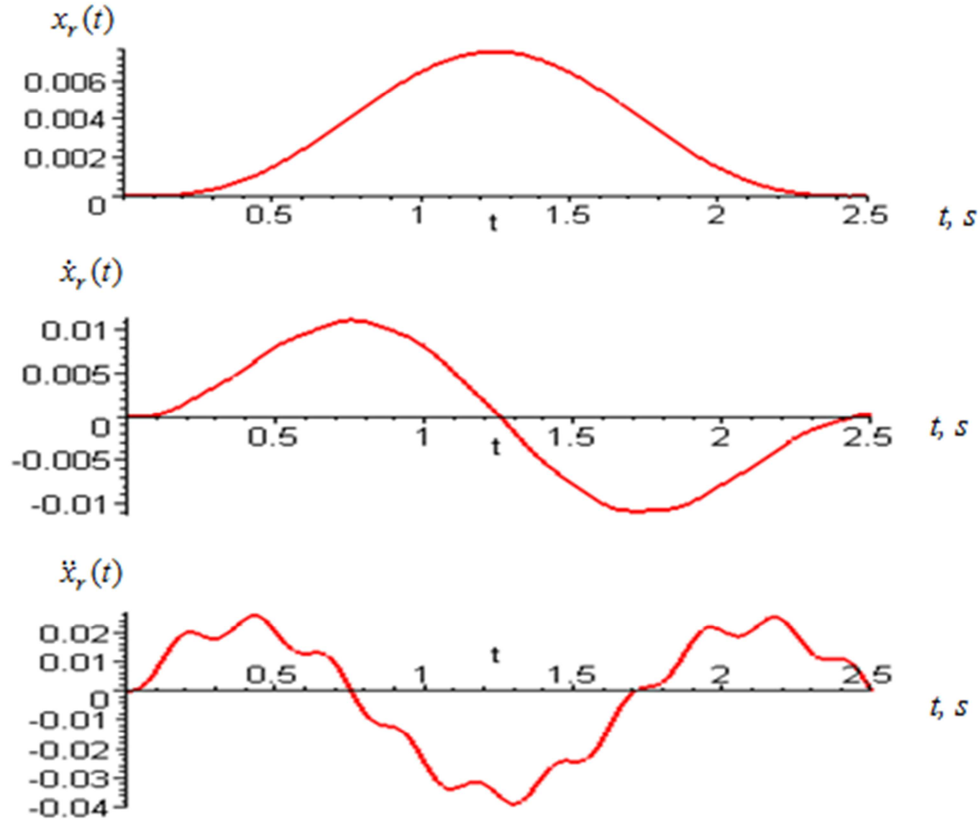

Fig.3. Schedules of relative motion of system with one degree of freedom

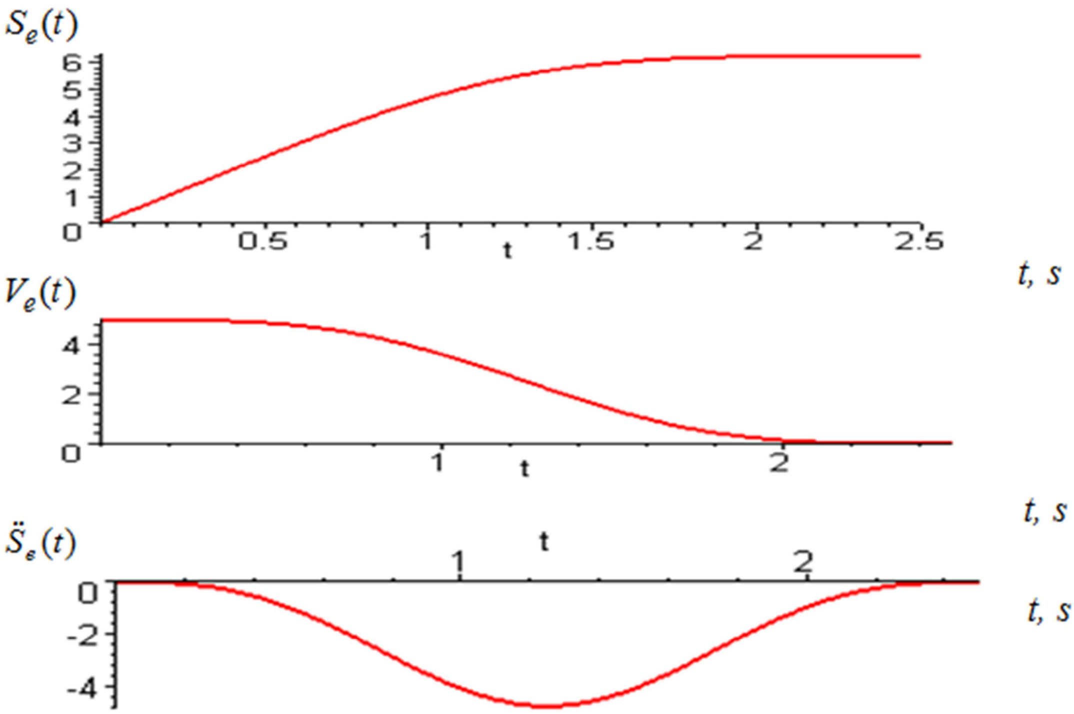

Fig.4. Schedules of figurative motion of system at optimum braking

Functional (9) is the criterion of difficult optimum motion of elastic system with one degree of freedom:

$$J = J_r + J_e = \int_0^T \left(\frac{\dot{x}_r^2}{2} - \frac{k^2 x_r^2}{2} - \ddot{S}_e x_r + \frac{\dot{S}_e^2}{2} - \ddot{S}_e \cdot S_e \right) dt. \quad (9)$$

The mathematical image of a functional (9) corresponds to the action principle in the form of Hamilton – Ostrogradsky and, naturally, Euler's equations for (9) will be:

$$\ddot{S}_e = a\sin^3(pt)\,, \qquad \ddot{x}_r + \omega^2 x_r = -\ddot{S}_e\,. \quad (10)$$

Found before the decisions (6) and (8) of the equations (10) satisfy to regional conditions: at the transfer motion $S_e(0) = 0$; $\dot{S}_e = V_e(0) = V_0$; $S_e(T) = L$; $\dot{S}_e = V_e(L) = 0$; at the relative motion – movement and speed at the initial and final moments of time are equal to zero. Optimum braking of object provides its motion into the final condition of absolute rest.

If frequency of own fluctuations k of elastic system with one degree of freedom (period $T_1 = 2\pi/k$) and the law of optimum control (braking) are known, after shutdown of control the fluctuations of object at the end of motion are absent ($t \geq T_*$).

3. A numerical example of elastic suspension braking (with 6 degrees of freedom)

For research of elastic suspension behavior the optimum figurative motion on the axis y under the law (6) is accepted; $k = \omega_1 = 151{,}7925$ [1/s]. Schedules of relative motion (acceleration and speed) and transfer motion are represented in figure 5.

Example. Basic data:
OM:=233: mass of the object.
XXI:=36.75:YYI:=35.87:ZZI:=13.1: physical inertial moments.
Coordinates of fastening points of shock-absorbers:
X:=[0.223,0.223,-0.207,-0.207,0.234,-0.22]:
Y:=[0.2,-0.19,-0.19,0.2,-0.281,-0.281]:
Z:=[-0.6235,-0.6235,-0.6235,-0.6235,0.617,0.617]:
Dynamic rigidities **CU:=[0.17*10^7,0.17*10^7,0.17*10^7,0.17*10^7,0.12*10^7,0.12*10^7]:**
CV:=[0.9*10^6,0.9*10^6,0.9*10^6,0.9*10^6,0.65*10^6,0.65*10^6]:
CW:=[0.22*10^7,0.22*10^7,0.22*10^7,0.22*10^7,0.11*10^7,0.11*10^7]:

Frequencies of own fluctuations of the elastic suspension, located on their increasing:

ω = 139.0505867; 151.7924930; 207.3618550; 247.9892090; 273.8719123; 355.8305395.

Schedules of relative and transfer motions of object along an axis y are provided below (figure 5).

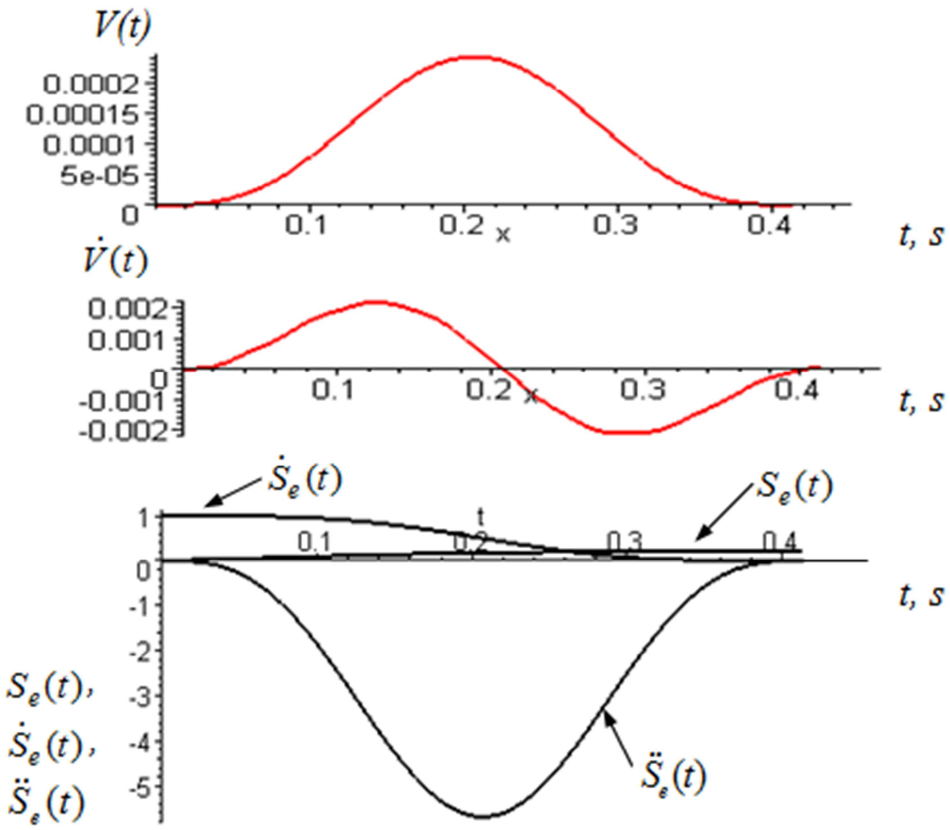

Fig.5. Schedules of motion of elastic suspension on an axis y

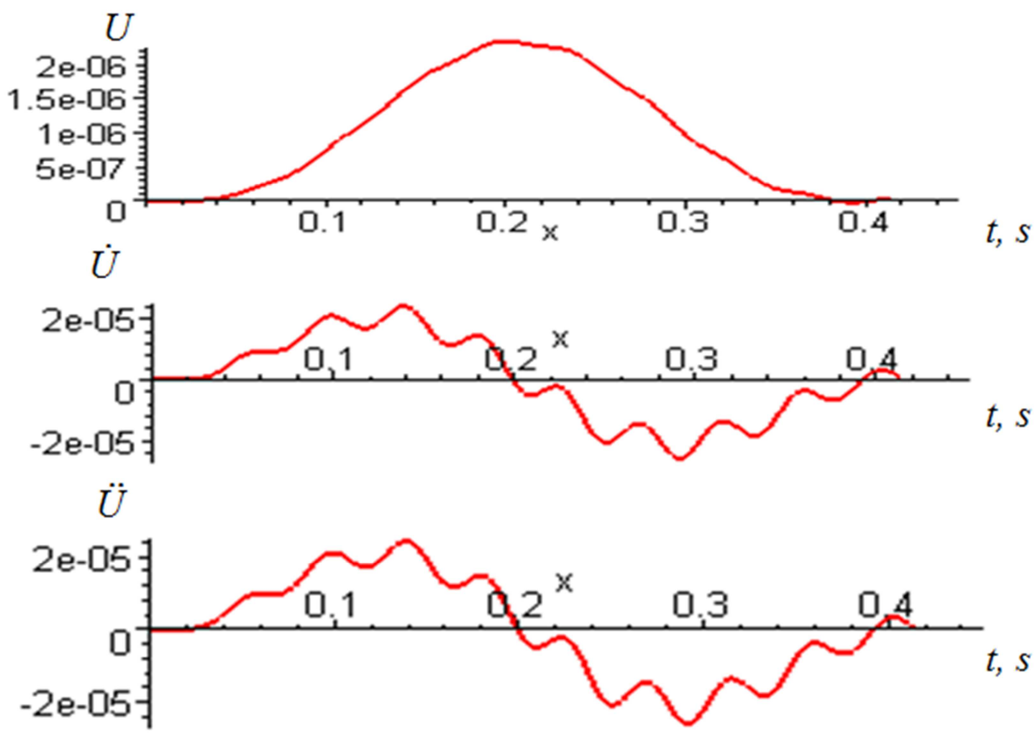

Fig.6. Schedules of relative motion of elastic suspension on an axis x

As appears from schedules, the absolute rest comes at the end of motion. The choice of time of motion (depending on the period of the main tone of fluctuations) and peak value of transfer acceleration provide elimination of fluctuations at the end of motion of elastic suspension.

Schedules of fluctuations of elastic suspension *on the axis x* at its optimum braking on an axis *y* are represented in figure 6. From schedules follows, that relative rest is observed in the axis *x* at the end of motion too.

The offered control model of braking at figurative motion of elastic object (suspension) can be used at optimum braking of the modern car. The considered approach at justification of optimum control by braking of elastic object extends on a rotary motion of large-sized elastic systems, in-cluding, for example, installation of nonrigid designs in a space [3, 4].

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